GIT APPLIED TO THE GEOMETRY OF LIE GROUPS - ELENA 2019

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The symmetries of any given geometric object form a group. This elementary fact is certainly much deeper than appears, the interplay between geometric properties of the object and algebraic properties of the group can be quite fruitful, particularly when symmetries abound. The *homogeneous* case, i.e. when such a group is large enough to act transitively on the object, is our leitmotiv. In many different kinds of geometries, including Riemannian, complex, symplectic and G_2, the role of invariant geometric structures on Lie groups and homogeneous spaces has always been very significant, both in inspiring some conjectures for the general case and providing counterexamples to some others. Needless to say, the interplay between geometry and symmetry is further enriched by the underlying topology.

Riemannian manifolds. Differentiable manifolds are often roughly described as topological spaces where one can do calculus very much like on $\mathbb{R}^n$ by just taking coordinates. A crucial fact that this presentation is missing is the strong use one makes in analysis of the canonical inner product on $\mathbb{R}^n$; actually, on each of its tangent spaces. Let us mention some of the limitations a differentiable manifold M has: the derivative of a vector field X on M at a point $p \in M$ with respect to a tangent vector $v \in T_pM$ is not well defined, in the sense that it depends on the curve passing through p with velocity v chosen; we can not measure the length of a tangent vector and so the length of a curve either; it is impossible to associate the gradient field or the Hessian to a smooth function $f : M \longrightarrow \mathbb{R}$.

All this can be considered as enough motivation to assign an inner product on each tangent space T_pM of a differentiable manifold M. A smooth choice is known as a *Riemannian metric*, a concept which does make geometry germinate in different ways: a distance on M defining the same topology, area or volume of regions, isometries, a connection or differentiation of vector fields with respect to tangent vectors, geodesics or (locally) shortest paths, parallel transportation along curves, holonomy, Laplace-Beltrami operator, and the most important one, *curvature*, a quite sophisticated concept that generalizes to higher dimensions the Gaussian curvature of surfaces in $\mathbb{R}^3$.

Lie groups. Consider a subgroup of matrices $G \subset \mathrm{GL}_n(\mathbb{R})$, which is therefore a *topological group*, in the sense that both multiplication and inversion are continuous functions. Remarkably, if in addition G is closed, then G can be endowed with a differentiable structure (same topology) such that multiplication and inversion are smooth, i.e. G is a *Lie group*. Examples other than $\mathrm{GL}_n(\mathbb{R})$ are $\mathrm{SL}_n(\mathbb{R})$ ($\det A = 1$), $\mathrm{O}(n)$ ($AA^t = I$), $\mathrm{GL}_n(\mathbb{C})$ ($A \in \mathrm{GL}_{2n}(\mathbb{R})$, $AJ = JA$, $J^2 = -I$), $\mathrm{U}(n) = \mathrm{GL}_n(\mathbb{C}) \cap \mathrm{O}(2n)$ and $\mathrm{Sp}(n, \mathbb{R})$ ($A \in \mathrm{GL}_{2n}(\mathbb{R})$, $A^tJA = J$). Lie groups show up across all the fields of mathematics as the set of symmetries of both algebraic structures and geometric objects and spaces. Since the set of matrices

$$\mathfrak{g} := \left\{ \alpha'(0) \in \mathfrak{gl}_n(\mathbb{R}) : \begin{array}{l} \alpha : (-\epsilon, \epsilon) \longrightarrow \mathfrak{gl}_n(\mathbb{R}) \text{ is smooth,} \\ \alpha(0) = I \text{ and } \alpha(t) \in G, \ \forall t \in (-\epsilon, \epsilon) \end{array} \right\},$$

is the tangent space of G at the identity matrix I, one obtains that $A\mathfrak{g}A^{-1} \subset \mathfrak{g}$ for any $A \in G$ by taking $\beta(t) = A\alpha(t)A^{-1}$. Another remarkable consequence of the fact that G is

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not only a closed subset but also a subgroup is that $\mathfrak{g}$ is closed under the usual bracket of matrices:

$$[X,Y] = XY - YX \in \mathfrak{g}, \qquad \forall X, Y \in \mathfrak{g}.$$

Indeed, this follows by differentiating at $t=0$ the curve $\alpha(t)Y\alpha(t)^{-1}$, where $\alpha(t) \in G$, $\alpha(0) = I$ and $\alpha'(0) = X$. $\mathfrak{g}$ is called the *Lie algebra* of G; for instance, the Lie algebras of the Lie groups mentioned above are respectively $\mathfrak{sl}_n(\mathbb{R})$ ($\operatorname{tr} X = 0$), $\mathfrak{so}(n)$ ($X^t = -X$), $\mathfrak{gl}_n(\mathbb{C})$ ($XJ = JX$), $\mathfrak{u}(n) = \mathfrak{gl}_n(\mathbb{C}) \cap \mathfrak{so}(2n)$ and $\mathfrak{sp}(n,\mathbb{R})$ ($X^tJ = -JX$).

The fact that the derivative of the exponential function $e : \mathfrak{gl}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R})$ at the zero matrix is the identity map implies that the subset $e^{\mathfrak{g}} \subset G$ contains a neighborhood of I in G. It follows that $e^{\mathfrak{g}}$ generates the connected component G_0 of G containing the identity I, with the following outcomes:

- $\mathfrak{g} = 0$ if and only if G is discrete;
- two closed subgroups with same Lie algebra must have identical connected components (e.g. $\mathrm{O}(n)$ and $\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R})$);
- if $\mathfrak{g}$ is *abelian* (i.e. $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$), then G_0 is a commutative group.

The topology of G is also affected by its Lie algebra $\mathfrak{g}$. For example, if $X^t = -X$, then $|e^{tX}| \equiv 1$, which implies that G_0 is compact when $\mathfrak{g} \subset \mathfrak{so}(n)$, and if on the contrary there is an $X \in \mathfrak{g}$ with a nonzero real eigenvalue, then clearly $e^{\mathbb{R}X}$ can not be bounded and G is non-compact. On the other hand, the derivative at the identity $d\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ of any smooth homomorphism $\varphi : G \longrightarrow H$ is a Lie algebra homomorphism such that $\varphi(e^X) = e^{d\varphi(X)}$ for all $X \in \mathfrak{g}$. Consequently, if $\varphi, \psi : G \longrightarrow H$ are two smooth homomorphisms such that $d\varphi = d\psi$, then $\varphi = \psi$ on G_0.

In conclusion, a closed subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ is not just any imbedded submanifold. The topology, geometry and structure of G are indeed strongly influenced by its Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$, a purely algebraic object.

Homogeneous spaces. Let M be the set of all 2-dimensional subspaces of $\mathbb{R}^n$. It is certainly not easy to figure out what would be a natural topology on M, and even harder, a natural differentiable structure and Riemannian metric. Is M 'bounded' or 'compact'? Can we 'continuously' go from a given subspace to any other one? What is the 'distance' (or 'shortest path') between two given 2-dimensional subspaces of $\mathbb{R}^n$?

However, the Lie group $\mathrm{O}(n)$ acts transitively on M and the isotropy subgroup at the subspace $\operatorname{span}\{e_1, e_2\}$ is $\mathrm{O}(2) \times \mathrm{O}(n-2)$, giving rise to the following natural presentation of M as a quotient topological space:

$$M = \mathrm{O}(n)/(\mathrm{O}(2) \times \mathrm{O}(n-2)).$$

Thus M is connected, it is compact and we can guess that M depends on $2(n-2)$ parameters by just subtracting dimensions. Indeed, M is a $2(n-2)$-dimensional differentiable manifold as is any quotient G/K of Lie groups with K a closed subgroup of G, called a *homogeneous space*. Moreover, if K is compact, then there exists a Riemannian metric on G/K such that G acts (transitively) by isometries. This certainly more than answers our original questions on the set M in a very natural way.

Note that any Lie group G is a homogeneous space acted by itself by left multiplication, so K is trivial in this case.

Geometric structures on Lie groups. Our main object of study is (G, γ), that is, a Lie group G endowed with a *left-invariant* geometric structure γ, e.g. a Riemannian metric, an almost-hermitian structure, a G_2-structure, etc. Thus γ is identified with a tensor (or set of tensors) on the Lie algebra $\mathfrak{g}$ of G and left-invariant means that $\gamma_p = dL_p|_e\gamma$ for any $p \in G$, where L_p is the diffeomorphism of G defined as left-multiplication by p. All that follows essentially works for G-invariant structures on a homogeneous space G/K, though a more technical exposition would be necessary.

The following natural questions and problems are our motivations:

- Is there any 'best' or 'canonical' γ on a given Lie group G? The meaning of the adjectives in the question is part of the problem.
- Existence and uniqueness of well-known special geometric structures on G, e.g. Einstein metrics, Kähler structures, geometric structures satisfying pinching curvature conditions, etc.
- The behavior of solutions to geometric flows of the form

$$\frac{d}{dt}\gamma(t) = q(\gamma(t)), \tag{1}$$

  where $q(\gamma)$ is a 'preferred' or 'optimal' tangent direction at each γ, e.g. the Ricci flow for Riemannian metrics.
- Given a notion of a preferred tangent direction q, it is natural to consider a geometric structure γ to be distinguished when $q(\gamma)$ is tangent to the equivalence class of γ (up to scaling). Heuristically, such a γ is in a sense nice enough that it can not be improved by the flow (1). These structures are called *solitons* and their existence, uniqueness, structure and classification give rise to challenging problems.

Both the answers and results related to all of the above questions are expected to be in terms of the algebraic structure of $\mathfrak{g}$ and the topology of G.

The moving-bracket approach. The fact that all the geometric information on (G, γ) is encoded in just γ and the Lie bracket μ of $\mathfrak{g}$ suggests the following viewpoint, which was used by J. Heber in the seminal article [H] and we have further developed in a series of papers, starting with [L01] and including [L06, L11, L12, L14, L16]. We consider the algebraic subset $\mathcal{L} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ of all Lie brackets on the vector space $\mathfrak{g}$ and fix a suitable tensor γ on $\mathfrak{g}$. Each $\mu \in \mathcal{L}$ is therefore identified with (G_μ, γ), the simply connected Lie group G_μ with Lie algebra $(\mathfrak{g}, \mu)$ endowed with the left-invariant geometric structure on G_μ defined by the fixed γ:

$$\mu \longleftrightarrow (G_\mu, \gamma). \tag{2}$$

The natural $\mathrm{GL}(\mathfrak{g})$-actions on tensors provide the following key equivalence between geometric structures:

$$(G_{h\cdot\mu}, \gamma) \simeq (G_\mu, h^*\gamma), \qquad \forall h \in \mathrm{GL}(\mathfrak{g}), \tag{3}$$

given by the Lie group isomorphism $G_{h\cdot\mu} \to G_\mu$ with derivative h^{-1}. This shows that, under identification (2), the isomorphism class $\mathrm{GL}(\mathfrak{g}) \cdot \mu$ contains all geometric structures of the same type of γ (up to equivalence) on the Lie group G_μ, for each $\mu \in \mathcal{L}$. Thus one has in $\mathcal{L}$, all together, all Lie groups of a given dimension endowed with left-invariant geometric structures of a given type. This perspective prepares the ground for the following useful strategies and facts:

- One can consider natural curvature functionals defined on the whole $\mathcal{L}$ and study their critical points, in order to provide good candidates for canonical structures (cf. [L01, L02, LW18]).
- Since any geometric quantity associated to (G_μ, γ) depends continuously on the Lie bracket μ, if μ *degenerates to* λ (i.e. $\lambda \in \overline{\mathrm{GL}(\mathfrak{g}) \cdot \mu}$) and G_λ admits a left-invariant geometric structure satisfying a strict pinching curvature condition, then there also must be a structure on G_μ for which the same pinching curvature condition holds (cf. [L03, NN, L16, DeL]).

- The usual convergence of a sequence of brackets produces the convergence of the corresponding geometric structures in well-known and natural senses like pointed (or Cheeger-Gromov) and smooth up to pull-back by diffeomorphisms, under suitable conditions (cf. [L11, L12, BL1]).
- Remarkably, a degeneration as above may give rise to the convergence of a sequence of geometric structures on a given Lie group toward a structure on a different Lie group, which might be non-homeomorphic (cf. [L11, L13, LW17, L17]).

The moving-bracket approach has been applied by several people in homogeneous geometry (cf. [L16, Section 5] and the references therein and [BL1]). In most of the applications, concepts and results from geometric invariant theory, including moment maps, closed orbits, stability, categorical quotients and Kirwan stratification, have been exploited in one way or another (see GIT section below).

The bracket flow. The role of (locally) homogeneous manifolds in Ricci flow theory has been very important, and in the last five years, Lie groups have played an even stronger role in the evolution of hermitian, symplectic and G_2 structures, due to the lack of examples one often faces in these geometries.

Provided by equivalence (3), our main tool to study a geometric flow as in (1) is a dynamical system defined on the variety of Lie algebras $\mathcal{L}$ called the *bracket flow*, introduced in [L11, L13]. The bracket flow is equivalent in a precise sense to (1) and it is defined by

$$\frac{d}{dt}\mu(t) = Q_{\mu(t)} \cdot \mu(t), \tag{4}$$

where $Q_\mu \in \mathfrak{gl}(\mathfrak{g})$ is a suitable operator such that $Q_\mu \cdot \gamma = q(G_\mu, \gamma)$. One has that,

- $\mu(t) \in \mathrm{GL}(\mathfrak{g}) \cdot \mu(0)$ for all t.
- The maximal interval of time where a solution exists is the same for both flows.
- The norm of the velocity of the flow must blow up at a finite-time singularity.

The approach is useful to better visualize the possible pointed limits of solutions, under diverse rescalings, as well as to address regularity issues. Immortal, ancient and self-similar solutions arise naturally from the qualitative analysis of the bracket flow.

Algebraic solitons. Due perhaps to its neat definition as a combination of geometric and algebraic aspects of (G, γ), the concept of algebraic soliton has a long and fruitful history since its introduction in [L01]. The following conditions are equivalent:

- γ is a soliton.
- $q(\gamma) = c\gamma + \mathcal{L}_X\gamma$ for some $c \in \mathbb{R}$, $X \in \mathfrak{X}(G)$.
- The solution $\gamma(t)$ to (1) starting at γ is self-similar, i.e. $\gamma(t) = c(t)f(t)^*\gamma$ for some $c(t) \in \mathbb{R}$ and $f(t) \in \mathrm{Diff}(G)$.

It is natural in our context to wonder about self-similar solutions evolving by automorphisms of G rather than just diffeomorphisms. We call (G_μ, γ) a *semi-algebraic soliton* if the solution $\gamma(t)$ to (1) starting at γ is given by $\gamma(t) = c(t)f(t)^*\gamma$ for some $c(t) \in \mathbb{R}$ and $f(t) \in \mathrm{Aut}(G)$. We now have the following equivalent conditions (see [LaL13]):

- (G_μ, γ) is a semi-algebraic soliton.
- $q(G_\mu, \gamma) = c\gamma + \mathcal{L}_{X_D}\gamma$ for some $c \in \mathbb{R}$, $D \in \mathrm{Der}(\mu)$, where X_D denotes the vector field on G_μ defined by the one-parameter subgroup of automorphisms of G_μ defined by D.
- $Q_\mu = cI + D + D^t$ for some $c \in \mathbb{R}$, $D \in \mathrm{Der}(\mathfrak{g})$.

If in addition in the last item $D^t \in \mathrm{Der}(\mathfrak{g})$, then (G_μ, γ) is called an *algebraic soliton*. These very distinguished structures are precisely the fixed points of the bracket flow (4) and are geometrically characterized among solitons as those with a diagonal evolution.

Geometric invariant theory. From the point of view of geometric invariant theory (GIT for short), any $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ is *unstable* for the natural $\mathrm{GL}(\mathfrak{g})$-action, i.e. $0 \in \overline{\mathrm{GL}(\mathfrak{g}) \cdot \mu}$. By following the lines of the case of reductive group actions on projective algebraic varieties over an algebraically closed field developed in [K], we obtained in [L07] that for each nonzero μ, there is a unique symmetric direction $\beta \in \mathfrak{gl}(\mathfrak{g})$, up to conjugation, 'most responsible' of the instability of the orbit $\mathrm{GL}(\mathfrak{g}) \cdot \mu$, in the sense that $e^{t\beta} \cdot \mu$ converges to zero as $t \to \infty$ faster than for any other direction. In that case, one says that μ belongs to the stratum $\mathcal{S}_\beta$ and this therefore defines a $\mathrm{GL}(\mathfrak{g})$-invariant stratification of $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, which has finitely many strata.

This gave rise to the introduction in [L07] of our main tool in the Riemannian case, the *beta operator* of $(G, \langle \cdot, \cdot \rangle)$. The beta operator satisfies certain estimate involving the Ricci curvature of $(G, \langle \cdot, \cdot \rangle)$, where equality holds if and only if some strong structural conditions on $\mathfrak{g}$ hold. If G is unimodular, then β acts as the mandatory Ricci operator (up to scaling) in order for $(G, \langle \cdot, \cdot \rangle)$ to be an expanding semi-algebraic soliton (cf. [BL1, LW18, BL3, L19]).

On the other hand, the moment map (or $\mathrm{GL}(\mathfrak{g})$-gradient map) from GIT for this representation is the $\mathrm{O}(\mathfrak{g})$-equivariant map

$$\mathrm{m} : \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} \longrightarrow \mathrm{sym}(\mathfrak{g}),$$

defined implicitly by

$$\langle \mathrm{m}(\mu), E \rangle = \tfrac{1}{|\mu|^2} \langle E \cdot \mu, \mu \rangle, \qquad \forall E \in \mathrm{sym}(\mathfrak{g}). \tag{5}$$

The $\mathbb{R}^*\mathrm{O}(\mathfrak{g})$-invariant functional

$$\mathrm{E} : \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} \longrightarrow \mathbb{R}, \qquad \mathrm{E}(\mu) := |\,\mathrm{m}(\mu)|^2,$$

has the following nice and useful properties (cf. [HSS, BL2]):

- μ is a critical point of E if and only if μ is a global minimum of $\mathrm{E}\,|_{\mathrm{GL}(\mathfrak{g})\cdot\mu}$, if and only if $\mathrm{m}(\mu) = cI + D$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mu)$.
- Inside any orbit $\mathrm{GL}(\mathfrak{g}) \cdot \mu$, the set of critical points of E is either empty or a single $\mathbb{R}^*\mathrm{O}(\mathfrak{g})$-orbit.
- Each stratum $\mathcal{S}_\gamma$ is the unstable manifold of the critical set $C_\beta := \{\lambda \in \mathrm{Crit}(\mathrm{E}) : \mathrm{m}(\lambda) \in \mathbb{R}^*\mathrm{O}(\mathfrak{g}) \cdot \beta\}$, in the sense that $\mu \in \mathcal{S}_\beta$ if and only if the negative gradient flow of E converges to a point in C_β.
- If $\mu \in \mathcal{S}_\beta$, then the closure $\overline{\mathrm{GL}(\mathfrak{g}) \cdot \mu}$ meets C_β in a single $\mathbb{R}^*\mathrm{O}(\mathfrak{g})$-orbit.

Curiously enough, in the Riemannian case, the main part of the Ricci operator of $(G_\mu, \langle \cdot, \cdot \rangle)$ is given by $\mathrm{m}(\mu)$. This fact has certainly allowed beautiful applications of GIT to the study of many problems related to Ricci curvature of homogeneous manifolds, including Einstein and Ricci soliton metrics, Ricci flow and Ricci negative metrics.

Note on bibliography. The present note is not a survey. The references are only intended to be a guide for the reader interested in some topic, they are far from complete. The year that appears in the labels of some of my articles in the bibliography corresponds to the first appearance of the paper in arXiv.

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